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Global bifurcation of polygonal relative equilibria for masses, vortices and dNLS oscillators

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ABSTRACT

Given a regular polygonal arrangement of identical objects, turning around a central object (masses, vortices or dNLS oscillators), this paper studies the global bifurcation of relative equilibria in function of a natural parameter (central mass, central circulation or amplitude of the oscillation). The symmetries of the problem are used in order to find the irreducible representations, the linearization and, with the help of a degree theory, the symmetries of the bifurcated solutions.

1. Introduction

Consider a polygonal arrangement of n identical objects turning in a plane, at constant angular speed, around a central object. These objects may be masses, following Newton's law of attraction, or point vortices, with Kirchoff's law, or nonlinear oscillators coupled to nearest neighbors in a finite circular lattice and a common phase.

A relative equilibrium for these problems is a stationary solution in rotating coordinates [9]. For each angular speed there is a regular polygonal relative equilibrium and an associated central quantity (mass, or circulation or amplitude of the oscillation) which is taken as a parameter.

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The purpose of this paper is to prove that for certain explicit values of this parameter there is a global bifurcation of relative equilibria with a specific symmetry. The tools for this study is representation theory and a simple version of the equivariant topological degree, studied in [7]. As a matter of fact, the reduction to irreducible representations gives a very clear picture of the symmetries involved and will be used in forthcoming papers on the bifurcation of periodic solutions for these problems.

There is a vast literature on the $(n+1)$-body problem, beginning with Maxwell's model for the Saturn rings. The point vortices problem has also attracted a good deal of research. This is usually done with a combination of numerical and explicit computations, where the symmetry is regarded as a nuisance. In the present paper we hope to show that these symmetries, when considered as a whole, facilitate instead the study.

In the rest of this introduction we shall set more precisely the problems. Then, in Section 2, we shall see how the symmetry of the problem forces the Hessian of the system of equations to have a special structure and we shall introduce a transformation which will bring the Hessian in a block-diagonal form, according to the different isotropy types. Afterwards, in Section 3, we shall state our bifurcation results, both local and global, giving solutions with specific symmetries. In Section 4, we shall complete the spectral analysis for the n-body and n-vortex problems. The final section is on the discrete NLS, which belongs to a somewhat different field of application but where a very similar analysis may be performed.

Among the papers listed in the bibliography, in particular [10,14,1,2,11,12] and their respective references, the paper closest to our results is [10] for the n-body and the n-vortex problems. These authors find the same critical values of the parameters and use a normal form analysis and numerical computations in order to prove local bifurcation results.

1.1. $(n+1)$-*vortex problem*

One of our purposes is to study relative equilibria of $n+1$ vortices with circulations $\mu_0 = \mu$ and $\mu_j = 1$ for $j \in \{1, \ldots, n\}$. Let $q_j(t) \in \mathbb{R}^2$ be the position of the vortices and $x_j(t) = e^{-\omega J t} q_j(t)$ be their position in rotating coordinates, with angular speed ω. Then, the dimensionless equations for relative equilibria with frequency ω are

$$\omega \mu_j x_j = \sum_{i=0\ (i \neq j)}^{n} \mu_i \mu_j \frac{x_j - x_i}{\|x_j - x_i\|^2},$$

where J is the canonical symplectic matrix.

1.2. $(n+1)$-*body problem*

Another of our purposes is to study relative equilibria of $n+1$ bodies in the plane, where the bodies have masses $\mu_0 = \mu$ and $\mu_j = 1$ for $j \in \{1, \ldots, n\}$. Let $x_j(t)$ be the position of the bodies in the plane and in rotating coordinates, as above with angular speed $(\omega)^{1/2}$. It is well known that, after the change of variables, the dimensionless equations for relative equilibria with frequency $(\omega)^{1/2}$ are

$$\omega \mu_j x_j = \sum_{i=0\ (i \neq j)}^{n} \mu_i \mu_j \frac{x_j - x_i}{\|x_j - x_i\|^3}.$$

1.3. *General problem*

Now, we will set a formulation generalizing both previous problems. Let x be the vector $(x_0, x_1, \ldots, x_n)^T$, where T denotes the transposed, and $\mathcal{M}$ the matrix $diag(\mu, 1, \ldots, 1)$. Our aim is to look for critical points of the potential

$$V_\alpha(x) = \omega\big(x^T\mathcal{M}x\big)/2 + \sum_{i<j}\mu_i\mu_j\phi_\alpha\big(\|x_j - x_i\|\big), \tag{1}$$

where $\phi_\alpha(x)$ satisfies $\phi'_\alpha(x) = -1/x^\alpha$ for $\alpha \in [1,\infty)$.

Since the potential V has gradient

$$\nabla_{x_j}V(x) = \omega\mu_j x_j - \sum_{i=0\ (i\neq j)}^{n}\mu_i\mu_j\frac{x_j - x_i}{\|x_j - x_i\|^{\alpha+1}},$$

then the critical points of V are the relative equilibria of the vortex problem for $\alpha = 1$, and of the body problem for $\alpha = 2$. Also, the case $\alpha \geqslant 1$ can be regarded as a problem of relative equilibria for bodies, where the general attraction potential is ϕ_α.

Hereafter we represent points in $\mathbb{R}^2$ and $\mathbb{C}$ indistinctly. Let $\zeta = 2\pi/n$ and let us set the positions of the bodies a_j as $a_0 = 0$ and $a_j = e^{ij\zeta}$ for $j \in \{1,\dots,n\}$. We see that a_j form a relative equilibrium with a central massive body at the origin surrounded by bodies of equal masses in a regular polygon. This polygonal relative equilibrium was studied by Maxwell as a simplified model of Saturn and its rings.

Proposition 1. *$\bar{a} = (a_0,\dots,a_n)$ is a critical point of the potential $V(x)$, when $\omega = \mu + s_1$ with*

$$s_1 = \sum_{j=1}^{n-1}\frac{1 - e^{ij\zeta}}{\|1 - e^{ij\zeta}\|^{\alpha+1}} = \frac{1}{2^\alpha}\sum_{j=1}^{n-1}\frac{1}{\sin^{(\alpha-1)}(j\zeta/2)}.$$

Proof. For $j = 0$, we have $\nabla_{x_0}V(\bar{a}) = \mu\sum_{j=0}^{n-1}e^{ij\zeta} = 0$. For $j \neq 0$, we have

$$\nabla_{x_j}V(\bar{a}) = \omega a_j - \sum_{i=1\ (i\neq j)}^{n}\frac{a_j - a_i}{\|a_j - a_i\|^{\alpha+1}} - \mu a_j = a_j\big(\omega - (\mu + s_1)\big).$$

Therefore, $\bar{a}$ is a relative equilibrium for frequencies $\omega = \mu + s_1$. $\square$

Notice that any homograph of a relative equilibrium is also a relative equilibrium. This is why we have decided to fix the norm of the relative equilibrium and leave the parameter μ free. Our objective is to find global bifurcation of relative equilibria from a_j using the parameter μ. Now let us see the symmetries of the problem.

Definition 2. Let S_n be the group of permutations of $\{1,\dots,n\}$ and let D_n be the subgroup generated by the permutations $\zeta(j) = j + 1$ and $\kappa(j) = n - j$. We define the action of S_n in $\mathbb{R}^{2(n+1)}$ as

$$\rho(\gamma)(x_0, x_1,\dots,x_n) = (x_0, x_{\gamma(1)},\dots,x_{\gamma(n)}).$$

In addition, we define the action of $O(2) = S^1 \cup \kappa S^1$ as

$$\rho(\theta) = e^{-\mathcal{J}\theta} \quad \text{and} \quad \rho(\kappa) = \mathcal{R},$$

where $\mathcal{R}$ is the matrix $diag(R,\dots,R)$ with $R = diag(1,-1)$.

Because n of the bodies have equal masses, the potential V is S_n-invariant. Moreover, the potential V is $O(2)$-invariant since the equations are invariant by rotating or reflecting the positions of all the bodies. Consequently, the gradient ∇V is Γ-equivariant with $\Gamma = S_n \times O(2)$. This means just that

$$\nabla V\big(\rho(\gamma)x\big) = \rho(\gamma)\nabla V(x)$$

for all $\gamma \in \Gamma$.

Let $\tilde{D}_n$ be the group generated by the elements (ζ,ζ) and (κ,κ) of $S_n \times O(2)$, where $\zeta = 2\pi/n \in S^1$. The action of (ζ,ζ) and (κ,κ) in $\mathbb{R}^{2(n+1)}$ is

$$(\zeta,\zeta)x = \rho(\zeta)e^{-\mathcal{J}\zeta}x \quad \text{and} \quad (\kappa,\kappa)x = \rho(\kappa)\mathcal{R}x.$$

As the action of (ζ,ζ) and (κ,κ) leaves the equilibrium $\bar{a}$ fixed, then its isotropy group, i.e. the subgroup of Γ which fixes the orbit $\bar{a}$, is

$$\Gamma_{\bar{a}} = \tilde{D}_n.$$

2. Irreducible representations

In order to prove the bifurcation theorem, we need to find the spaces of irreducible representations of $\tilde{D}_n$.

Let us define A_{ij} to be the 2×2 submatrices of $D^2V(\bar{a})$ such that

$$D^2V(\bar{a}) = A = (A_{ij})_{ij=0}^{n}.$$

Due to the fact that $D^2V(\bar{a})$ is $\tilde{D}_n$-equivariant, one has the following result:

Proposition 3. *The blocks A_{ij} satisfy the relations*

$$A_{ij} = e^{-J\zeta}A_{\zeta(i)\zeta(j)}e^{J\zeta} \quad \textit{and} \quad A_{ij} = RA_{\kappa(i)\kappa(j)}R. \tag{2}$$

Proof. Since the matrix $D^2V(\bar{a})$ is $\tilde{D}_n$-equivariant, then the matrix A and $\rho(\zeta)e^{-\mathcal{J}\zeta}$ commute. Therefore,

$$A = \rho(\zeta)e^{-\mathcal{J}\zeta}Ae^{\mathcal{J}\zeta}\rho(\zeta)^{-1}.$$

Hereafter, we denote by $[u]_i$ the coordinate $u_i \in \mathbb{R}^2$ of the vector $u = (u_0,\ldots,u_n)^T$. Therefore,

$$\begin{aligned}\big[\rho(\zeta)e^{-\mathcal{J}\zeta}Ae^{\mathcal{J}\zeta}\rho(\zeta)^{-1}u\big]_i &= e^{-J\zeta}\big[Ae^{\mathcal{J}\zeta}\rho(\zeta)^{-1}u\big]_{\zeta(i)} \\ &= e^{-J\zeta}\sum A_{\zeta(i)j}\big(e^{J\zeta}\big)u_{\zeta^{-1}(j)} = \sum e^{-J\zeta}A_{\zeta(i)\zeta(j)}e^{J\zeta}u_j.\end{aligned}$$

From this equality, we get that

$$\sum_j A_{ij}u_j = [Au]_i = \sum e^{-J\zeta}A_{\zeta(i)\zeta(j)}e^{J\zeta}u_j.$$

Then we conclude that $A_{ij} = e^{-J\zeta}A_{\zeta(i)\zeta(j)}e^{J\zeta}$. Using a similar argument and the fact that A and $\rho(\kappa)\mathcal{R}$ commute, we obtain $A_{ij} = RA_{\kappa(i)\kappa(j)}R$. $\square$

Now, we may find the irreducible representations of the action of the group $\tilde{D}_n$.

Since the irreducible representations are different for $n=2$ and $n \geqslant 3$, we shall concentrate of the case $n \geqslant 3$ in the remaining part of the paper, except for comments on the case $n=2$.

Definition 4. Let us define the vectors v_1 and v_{n-1} as

$$v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{and} \quad v_{n-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

For $k \in \{2, \ldots, n-2, n\}$, we define the isomorphisms $T_k : \mathbb{C}^2 \to V_k$ as

$$T_k(z) = \left(0, n^{-1/2} e^{(ikI+J)\zeta} z, \ldots, n^{-1/2} e^{n(ikI+J)\zeta} z\right) \quad \text{with}$$
$$V_k = \left\{\left(0, e^{(ikI+J)\zeta} z, \ldots, e^{n(ikI+J)\zeta} z\right) \colon z \in \mathbb{C}^2\right\},$$

and for $k \in \{1, n-1\}$, we define the isomorphism $T_k : \mathbb{C}^3 \to V_k$ as

$$T_k(\alpha, w) = \left(v_k \alpha, n^{-1/2} e^{(ikI+J)\zeta} w, \ldots, n^{-1/2} e^{n(ikI+J)\zeta} w\right) \quad \text{with}$$
$$V_k = \left\{\left(v_k \alpha, e^{(ikI+J)\zeta} w, \ldots, e^{n(ikI+J)\zeta} w\right) \colon w \in \mathbb{C}^2,\ \alpha \in \mathbb{R}\right\}.$$

Next let us find the action of the group $\tilde{D}_n$ on the subspaces V_k.

Proposition 5. *The actions of* (ζ, ζ) *and* (κ, κ) *on* V_k *are*

$$(\zeta, \zeta) T_k(z) = T_k\left(e^{ik\zeta} z\right) \quad \textit{and}$$
$$(\kappa, \kappa) T_k(z) = T_{n-k}(Rz),$$

where R is the matrix diag$(1, 1, -1)$ *for the special cases* $k \in \{1, n-1\}$.

Proof. For $k \in \{2, \ldots, n-2, n\}$, we have

$$\left[\rho(\zeta) T_k(z)\right]_j = \left[T_k(z)\right]_{\zeta(j)} = n^{-1/2} e^{j(ikI+J)\zeta}\left(e^{(ikI+J)\zeta} z\right)$$
$$= \left[T_k\left(e^{(ikI+J)\zeta} z\right)\right]_j.$$

Therefore $\rho(\zeta) T_k(z) = T_k(e^{(ikI+J)\zeta} z)$. Since the element $\zeta \in O(2)$ acts as $e^{-\zeta \mathcal{J}} T_k(z) = T_k(e^{-\zeta J} z)$, we conclude that (ζ, ζ) acts as $(\zeta, \zeta) T_k(z) = T_k(e^{ik\zeta} z)$.

For $k \in \{1, n-1\}$ we have, as before, that

$$\rho(\zeta) T_k(\alpha, w) = T_k\left(\alpha, e^{(ikI+J)\zeta} w\right).$$

Moreover, from the equality $e^{-J\zeta} v_k = e^{ik\zeta} v_k$ we get that $\zeta \in O(2)$ acts as

$$e^{-\zeta \mathcal{J}} T_k(\alpha, w) = T_k\left(e^{ik\zeta} v_k \alpha, e^{-J\zeta} w\right).$$

Hence, the action in this case is also $(\zeta, \zeta) T_k(z) = T_k(e^{ik\zeta} z)$.

It remains to find the action of (κ,κ). For $k \in \{2,\dots,n-2,n\}$, we have

$$\begin{aligned}\big[(\kappa,\kappa)T_k(z)\big]_j &= \big[\mathcal{R}T_k(z)\big]_{\kappa(j)} = n^{-1/2}e^{j(-ikI+J)\zeta}Rz\\ &= \big[T_{n-k}(Rz)\big]_j,\end{aligned}$$

therefore the action is $(\kappa,\kappa)T_k(z) = T_{n-k}(Rz)$. For $k \in \{1, n-1\}$, by a similar argument and the fact that $Rv_k = v_{\kappa(k)}$, we prove that the action is as before but with $R = diag(1,1,-1)$. □

Consequently, we have that the spaces $V_k \oplus V_{n-k}$ are subrepresentations of the action of $\tilde{D}_n$. Moreover, the action of (ζ,ζ) and (κ,κ) on the subspace $V_k \oplus V_{n-k}$ is

$$\begin{aligned}(\zeta,\zeta)(z_k, z_{n-k}) &= \big(e^{ik\zeta}z_k, e^{-ik\zeta}z_{n-k}\big) \quad \text{and}\\ (\kappa,\kappa)(z_k, z_{n-k}) &= (Rz_{n-k}, Rz_k).\end{aligned}$$

Let $\tilde{\mathbb{Z}}_n$ be the group generated by (ζ,ζ). Since the subspaces V_k are irreducible representations of $\tilde{\mathbb{Z}}_n$, by Schur's lemma (that is a linear map which commutes with action, must send equivalent representations into themselves), we obtain $D^2V(\bar{a})T_k(z) = T_k(B_k z)$. Furthermore, as $D^2V(\bar{a})$ commutes with the action of (κ,κ), then the blocks B_k must satisfy $B_kR = RB_{n-k}$. Consequently, there must be a map that puts the matrix $D^2V(\bar{a})$ in diagonal form with the blocks B_k. Clearly, the isomorphisms T_k, with range V_k, are the components of this orthogonal transformation.

Proposition 6. *Define the map $Pz = \sum_{k=1}^n T_k(z_k)$ for $z = (z_1,\dots,z_n)$, then the linear map P is orthogonal $P^* = P^{-1}$.*

Proof. Since the matrix $e^{jJ\zeta}$ is an isometry in $\mathbb{C}^2$ and

$$\sum_{j=0}^{n-1} e^{ij(k-l)\zeta} = n\delta_{kl},$$

for $k,l \in \{2,\dots,n-2,n\}$, then

$$\big\langle T_k(z_k), T_l(z_l)\big\rangle = n^{-1}\sum_{j=1}^{n} e^{ij(k-l)\zeta}\big\langle e^{jJ\zeta}z_k, e^{jJ\zeta}z_l\big\rangle = \delta_{kl}\langle z_k, z_l\rangle.$$

In fact, since v_1 and v_{n-1} are orthonormal vectors, one proves that $\langle T_k(z_k), T_l(z_l)\rangle = \delta_{kl}\langle z_k, z_l\rangle$ for all k and l. Therefore, the map P satisfies

$$\langle Pz, Pz\rangle = \sum \delta_{kl}\langle z_k, z_l\rangle = \langle z, z\rangle.$$

Thus, P is an isometry on $\mathbb{C}^{2(n+1)}$ and $P^* = P^{-1}$. □

By Schur's lemma, the matrix $D^2V(\bar{a})$ is diagonal in the new coordinates, that is

$$P^{-1}D^2V(\bar{a})P = diag(B_1,\dots,B_n).$$

Our next objective consists in finding the blocks B_k in terms of the matrices A_{ij}. Remember that the matrices A_{ij} are the 2×2 submatrices of the Hessian $D^2V(\bar{a})$.

Proposition 7. *For $k \in \{2, \ldots, n-2, n\}$ the blocks B_k are*

$$B_k = \sum_{j=1}^{n} A_{nj} e^{j(ikI+J)\zeta}.$$

Proof. For $l \neq 0$ we have $[AT_k(z)]_l = n^{-1/2} \sum_{j=1}^{n} A_{lj} e^{j(ikI+J)\zeta} z$. Now, from the relation (2), we prove that $A_{lj} = e^{lJ\zeta} A_{n(j-l)} e^{-lJ\zeta}$, with $l-j$ modulo n. Hence

$$\left[AT_k(z)\right]_l = n^{-1/2} \sum_{j=1}^{n} e^{l(ik+J)\zeta} A_{n(j-l)} e^{(j-l)(ikI+J)\zeta} z.$$

Consequently, we rewrite the sum as

$$\left[AT_k(z)\right]_l = n^{-1/2} e^{l(ik+J)\zeta} \left[\sum_{j=1}^{n} A_{nj} e^{j(ikI+J)\zeta} z \right] = \left[T_k(B_k z)\right]_l.$$

But since the isomorphisms T_k are defined on the invariant subspaces V_k, then $[AT_k(z)]_0 = [T_k(B_k z)]_0$ and we conclude that $AT_k(z) = T_k(B_k z)$. Actually, one may prove directly that $[AT_k(z)]_0 = [T_k(B_k z)]_0$, for instance see [4]. $\square$

Proposition 8. *For $k \in \{1, n-1\}$ the blocks B_k are*

$$B_k = \begin{pmatrix} e_1^T A_{00} e_1 & n^{1/2} \overline{(A_{n0} v_k)}^T \\ n^{1/2} A_{n0} v_k & \sum_{j=1}^{n} A_{nj} e^{j(ikI+J)\zeta} \end{pmatrix}.$$

Proof. For $l \neq 0$, we have

$$\left[AT_k(\alpha, w)\right]_l = (A_{l0} v_k)\alpha + n^{-1/2} \sum_{j=1}^{n} A_{lj} e^{j(ikI+J)\zeta} w,$$

where $\alpha \in \mathbb{C}$ and $w \in \mathbb{C}^2$. Now, from the symmetries (2) we prove that $A_{l0} = e^{lJ\zeta} A_{n0} e^{-lJ\zeta}$. Hence, $A_{l0} v_k = e^{lJ\zeta} A_{n0} e^{-lJ\zeta} v_k$. Moreover, since $e^{-lJ\zeta} v_k = e^{lik\zeta} v_k$, then $A_{l0} v_k = e^{l(ki+J)\zeta} A_{n0} v_k$. Using the previous computation, we find that

$$\left[AT_k(\alpha, w)\right]_l = n^{-1/2} e^{l(ikI+J)\zeta} \left[\left(n^{1/2} A_{n0} v_k\right)\alpha + \left(\sum_{j=1}^{n} A_{nj} e^{j(ikI+J)\zeta} \right) w \right]. \tag{3}$$

For $l = 0$, we have $[AT_k(z)]_0 = (A_{00} v_k)\alpha + n^{-1/2} D_k w$, with $D_k = \sum_{j=1}^{n} A_{0j} e^{j(ikI+J)\zeta}$. From the relations (2), we have that $A_{00} = cI$. Now, since $v_k \bar{v}_k^T = 1$, then

$$\left[AT_k(\alpha, w)\right]_0 = v_k\left[\left(e_1^T A_{00} e_1\right)\alpha + n^{-1/2}\left(\bar{v}_k^T D_k\right) w\right]. \tag{4}$$

Consequently, from the equalities (3) and (4), we obtain $D^2 V(\bar{a}) T_k(\alpha, w) = T_k(B_k(\alpha, w))$, with

$$B_k = \begin{pmatrix} e_1^T A_{00} e_1 & n^{-1/2} (\bar{v}_k^T D_k) \\ n^{1/2} A_{n0} v_k & \sum_{j=1}^{n} A_{nj} e^{j(ikI+J)\zeta} \end{pmatrix}.$$

Moreover, since the map P is orthonormal and the matrix A is self-adjoint, then B_k must be self-adjoint and $\overline{(\bar{v}_k^T D_k)}^T = nA_{n0}v_k$. Actually, one may prove directly that $nA_{n0}v_k = \overline{(\bar{v}_k^T D_k)}^T = D_k^T v_k$, for instance see [4]. □

Remark 9. In the computation of the blocks B_k, we have used only the symmetries of $D^2V(\bar{a})$. This will enable us to apply these results to a wide class of problems, as the dNLS equations at the final section. Also, notice that the change of variables was done in complex coordinates, and these will allow us to prove bifurcation of periodic solutions in a series of forthcoming papers analogous to [5]: as a matter of fact, the natural approach to the study of periodic solutions is, in this context, the use of Fourier series. Thus, the change of variables, which we have introduced, will be helpful.

However, in order to find bifurcation of relative equilibria, we need the change of variables for real coordinates.

Proposition 10. *If the matrix $D^2V(\bar{a})$ has domain $\mathbb{R}^{2(n+1)}$, then the matrix $P^{-1}D^2V(\bar{a})P$ has domain $W = P^{-1}\mathbb{R}^{2(n+1)}$ and*

$$P^{-1}D^2V(\bar{a})P = diag(B_1, B_2, \ldots, B_{n/2}, B_n) \quad with$$
$$W = \mathbb{C}^3 \times \mathbb{C}^2 \times \cdots \times \mathbb{R}^2 \times \mathbb{R}^2.$$

Moreover, the action on the block B_k is

$$(\zeta, \zeta)z_k = e^{ik\zeta}z_k \quad and \quad (\kappa, \kappa)z_k = R\bar{z}_k,$$

where R is the matrix $diag(1, 1, -1)$ when $k = 1$ and $diag(1, -1)$ for the remaining cases.

Proof. First, we need to identify the subspace $W = \{z\colon\ Pz \in \mathbb{R}^{2(n+1)}\}$. If Pz is real, then

$$\sum_{k=1}^{n} T_k(z_k) = Pz = \overline{Pz} = \sum_{k=1}^{n} T_{n-k}(\bar{z}_k).$$

Thus, the subspace W is the set of points $(z_1, \ldots, z_n)$ such that $z_{n-k} = \bar{z}_k$.

Remember that (κ, κ) acts on the coordinate z_k as $(\kappa, \kappa)z_k = Rz_{n-k}$. Hence, for $k \in \{n/2, n\}$ we have $z_k = z_{n-k} = \bar{z}_k \in \mathbb{R}^2$, then (κ, κ) acts as $(\kappa, \kappa)z_k = Rz_k$. Consequently, the blocks $B_{n/2}$ and B_n are defined in a real space with real action.

Now for $k \notin \{n/2, n\}$, we can take the isomorphism $T(z_k) = (z_k, z_{n-k})$ with $z_{n-k} = \bar{z}_k$. In this way (κ, κ) acts as

$$(\kappa, \kappa)T(z_k) = (R\bar{z}_k, Rz_k) = T(R\bar{z}_k).$$

Moreover, since (ζ, ζ) acts as $(\zeta, \zeta)z_k = e^{ik\zeta}z_k$, then $(\zeta, \zeta)T(z_k) = T(e^{ik\zeta}z_k)$. Finally, we use the equality $B_{n-k} = \bar{B}_k$ to prove that $(B_k, B_{n-k})T(z_k) = T(B_k z_k)$. □

Remark 11. If $n = 2$ we have to define T_2 as before, from $\mathbb{C}^2$ into V_2.

However, for $k = 1$, define the isomorphism $T_1 : \mathbb{C}^4 \to V_1$ as

$$T_1(v, w) = \left(v, 2^{-1/2}w, 2^{-1/2}w\right) \quad \text{with}$$
$$V_1 = \left\{(v, w, w)\colon\ v, w \in \mathbb{C}^2\right\}.$$

Then, the action of $\tilde{D}_2$ on $z_k \in V_k$ is

$$(\zeta,\zeta)z_2 = z_2 \quad \text{and} \quad (\kappa,\kappa)z_2 = Rz_2,$$
$$(\zeta,\zeta)z_1 = -z_1 \quad \text{and} \quad (\kappa,\kappa)z_1 = diag(R,R)z_1.$$

Hence, the spaces V_k are irreducible for the action of (ζ,ζ), but V_1 contains two representations, one where (κ,κ) acts as the identity and the other where this element acts minus the identity.

One may prove that A_{ij} are diagonal matrices and satisfy $A_{11} = A_{22}$, $A_{21} = A_{12}$ and $A_{01} = A_{10} = A_{02} = A_{20}$. Thus,

$$A_{00} = (s_1+\mu)\mu I - 2A_{20}, \qquad A_{20} = -\mu diag(\alpha,-1),$$
$$A_{21} = -\frac{1}{2^{\alpha+1}} diag(\alpha,-1) \quad \text{and} \quad A_{22} = (s_1+\mu)I - (A_{20}+A_{21}).$$

In particular, contrary to the case $n \geqslant 3$, where A_{00} is a multiple of the identity, this matrix is only diagonal.

The transformation P is orthogonal and one sends the Hessian into $diag(B_1, B_2)$, with $B_2 = (\alpha+1)(\mu+s_1)diag(1,0)$, but B_1 is now

$$\begin{pmatrix} \mu(s_1+\mu+2\alpha) & 0 & -\sqrt{2}\alpha\mu & 0 \\ 0 & \mu(s_1+\mu-2) & 0 & \sqrt{2}\mu \\ -\sqrt{2}\alpha\mu & 0 & s_1+(\alpha+1)\mu & 0 \\ 0 & \sqrt{2}\mu & 0 & s_1 \end{pmatrix}.$$

In order to study the bifurcation of relative equilibria we need to restrict the block B_1 to the fixed point subspace of (κ,κ), that is to the first and third coordinates. There,

$$B_1|_{V_1^{(\kappa,\kappa)}} = \begin{pmatrix} \mu(s_1+\mu+2\alpha) & -\sqrt{2}\alpha\mu \\ -\sqrt{2}\alpha\mu & s_1+(\alpha+1)\mu \end{pmatrix}.$$

The determinant of this matrix is $\mu(\mu+s_1)(2\alpha+\mu+s_1+\alpha\mu)$. At $\mu = 0, -s_1$, there will be a bifurcation, as in the case $n \geqslant 3$, and at

$$\mu_1 = -(2\alpha+s_1)/(\alpha+1)$$

there will be a bifurcation with symmetry $\tilde{D}_1$. For the vortices, one has $\alpha = 1$, $s_1 = 1/2$ and $\mu_1 = -5/4$, while, for the masses, one has $\alpha = 2$, $s_1 = 1/4$ and $\mu_1 = -17/12$.

3. Bifurcation theorem

We shall now give sufficient conditions for the bifurcation of relative equilibria from $\bar{a}$. To carry on this proof, we need to apply the change of variables P directly for the potential V.

In this manner, we define the potential $V_P : W \to \mathbb{R}$ as $V_P(x) = V(Px)$. Then the potential V_P is Γ-invariant and the gradient ∇V_P is Γ-equivariant with the action $\rho_P(\gamma) = P^{-1}\rho(\gamma)P$. Note that $x_0 = P^{-1}\bar{a}$ is the relative equilibrium in the new coordinates.

To find the symmetries, for each h dividing n, we define the group $\tilde{D}_h$ as the one generated by the elements $(n/h)(\zeta,\zeta)$ and (κ,κ). These groups $\tilde{D}_h$ are subgroups of the isotropy group $\tilde{D}_n$. Our approach consists in applying Brouwer degree to the maps $\nabla V_P(x)$ restricted to the spaces of fixed points of $\tilde{D}_h$. As seen in [7], this is equivalent to the $\tilde{D}_n$-equivariant degree.

So we set the function $f_h(x)$ as

$$f_h(x) = \nabla V_P(x)|_{W^{\tilde{D}_h}} : W^{\tilde{D}_h} \to W^{\tilde{D}_h}.$$

Then, the zeros of $f_h(x)$ are the relative equilibria with symmetry $\tilde{D}_h$. Now, the polygonal relative equilibrium is $x_0 = P^{-1}\bar{a}$, so x_0 is a zero of $f_h(x)$. Since we wish to prove existence of bifurcation from x_0, we need the sign of $\det f_h'(x_0)$.

Proposition 12. *Define σ_k as*

$$\sigma_k = sgn\big(e_1^T B_k e_1\big) \quad \textit{for } k \in \{n, n/2\} \quad \textit{and}$$
$$\sigma_k = sgn(\det B_k) \quad \textit{for } k \in [1, n/2) \cap \mathbb{N}. \tag{5}$$

Then

$$sgn\big(\det f_h'(x_0)\big) = n(\mu) = \sigma_n \prod_{j \in [1,n/2] \cap \mathbb{N}h} \sigma_j.$$

Proof. Since (ζ, ζ) acts on the coordinate z_k as $e^{ik\zeta} z_k$, the action of $(n/h)(\zeta, \zeta)$ on z_k is trivial when $z_k = e^{ik(2\pi/h)} z_k$. This happens precisely for the coordinates z_k with $k \in h\mathbb{N}$. Now, the action of (κ, κ) on the coordinate z_k is trivial whenever $z_k = R\bar{z}_k$. Therefore, $z \in W^{\tilde{D}_h}$ only when $z = (z_h, z_{2h}, \ldots, z_n)$ with $z_k = R\bar{z}_k$.

Then, the matrix $D^2 V_P(x_0)$, on the space $W^{\tilde{D}_h}$, is

$$diag(D_h, D_{2h}, \ldots, D_n),$$

where the blocks D_h are as follows:

- For $k \in \{n, n/2\}$, we have that $D_k = e_1^T B_k e_1$ because $z_h \in \mathbb{R} \times \{0\}$.
- For $k \notin \{1, n/2, n\}$, since $z_h \in \mathbb{R} \times i\mathbb{R}$, we have that $D_k = T^* B_k T$, where $T = diag(1, i)$ is the natural isomorphism between $\mathbb{R}^2$ and $\mathbb{R} \times i\mathbb{R}$.
- For $k = 1$, since $z_h \in \mathbb{R}^2 \times i\mathbb{R}$, we have $D_k = T^* B_k T$, where $T = diag(1, 1, i)$ is the natural isomorphism between $\mathbb{R}^3$ and $\mathbb{R}^2 \times i\mathbb{R}$.

Finally, from the definition of σ_k, we get that $sgn(\det D_k) = \sigma_k$ and therefore

$$sgn \det f_p'(x_0) = sgn\Bigg(\det D_n \prod_{j \in [1,n/2] \cap \mathbb{N}h} \det D_j\Bigg) = n(\mu). \qquad \square$$

Hence, we have given the sign of $\det f_h'(x_0)$ in terms of the blocks B_k.

3.1. Local bifurcation

In order to apply Brouwer degree and prove bifurcation, let us define f, from $B_{2\varepsilon} \times B_{2\rho}$ to $\mathbb{R} \times W$, as

$$f(x, \mu) = \big(\|x - x_0\| - \varepsilon, f_h(x, \mu)\big) \quad \text{with}$$
$$B_{2\varepsilon} \times B_{2\rho} = \big\{(x, \mu) \in W^{\tilde{D}_h} \times \mathbb{R}\colon \|x - x_0\| \leqslant 2\varepsilon,\ |\mu - \mu_0| \leqslant 2\rho\big\}.$$

Theorem 13. *The Brouwer degree of $f(x,\mu)$ is well defined and*

$$\deg(f; B_{2\varepsilon}\times B_{2\rho}) = \eta_h(\mu_0) = n_h(\mu_0-\rho) - n_h(\mu_0+\rho).$$

Hence, when $\eta_h(\mu_0)\neq 0$, there is a local bifurcation from (x_0,μ_0) with symmetry $\tilde{D}_h$.

Proof. As in [6], the proof consists in a linear deformation of the function $\|x-x_0\|-\varepsilon$ to $\rho-\|\mu-\mu_0\|$ and of the function $f_h(x)$ to $f_h'(x-x_0)$. Then, we may use the excision property to prove that

$$\deg\bigl(f(\mu); B_{2\rho}\times B_{2\varepsilon}\bigr) = \deg\bigl(f_h'(\mu_0-\rho)(x-x_0); B_{2\varepsilon}\bigr) - \deg\bigl(f_h'(\mu_0+\rho)(x-x_0); B_{2\varepsilon}\bigr).$$

Therefore, the first part of the proof follows from the fact that

$$\deg\bigl(f_h'(\mu)(x-x_0); B_{2\varepsilon}\bigr) = n_h(\mu).$$

Now, supposing $\eta_h(\mu_0)\neq 0$, for small ε there is an $(x_\varepsilon,\mu_\varepsilon)$ with $x_\varepsilon\subset W^{\tilde{D}_h}$ such that $f_h(x_\varepsilon,\mu_\varepsilon)=0$ and $d(x_\varepsilon,x_0)=\varepsilon$. Moreover, when we let ε tend to zero, by the compactness we have a series $\varepsilon_k\to 0$ such that $\mu_{\varepsilon_k}\to\mu_1$. By the continuity we conclude that $\mu_1=\mu_0$. □

When only one of the blocks B_k has a determinant which changes sign, we have the following result.

Theorem 14. *For $k\in\{1,\ldots,[n/2],n\}$, let h be the maximum common divisor of k and n. Supposing $\sigma_k(\mu)$ changes sign at μ_0 and $\sigma_j(\mu_0)\neq 0$ for the others $j\in[1,n/2]\cap(\mathbb{N}h)$, then there is a bifurcation with maximal symmetry $\tilde{D}_h$. This means that the local bifurcation is in $W^{\tilde{D}_h}\setminus\bigcup_{\tilde{D}_h\subset H}W^H$.*

Proof. To assure all the symmetries of the bifurcation, we apply the previous theorem with h the maximum common divisor of k and n. By hypothesis the product

$$\sigma_n \prod_{j\in[1,n/2]\cap\mathbb{N}h\ (j\neq k)} \sigma_j(\mu_0)$$

is not zero, then $\eta_h(\mu_0)=\pm 2$. Henceforth, there is a bifurcation in the fixed point space of $\tilde{D}_h$.

It only remains to prove that $\tilde{D}_h$ is the maximum group of symmetries. Let $\tilde{D}_p$ be a group such that $\tilde{D}_h\subset\tilde{D}_p$, this means that h divides p. Since p does not divide k, then

$$sgn \det f_p'(x_0) = \sigma_n \prod_{j\in[1,n/2]\cap\mathbb{N}p} \sigma_j \neq 0.$$

Consequently, the linear map $f_p'(x_0)$ is invertible, and by the implicit function theorem, we deduce the nonexistence of solutions in $W^{\tilde{D}_p}$ near (x_0,μ_0). □

3.2. Global bifurcation

Now, we wish to prove a global bifurcation result, which is just an adaptation of the Rabinowitz alternative. Notice that this approach may not give all the best information available for a global result, as an application of the Γ-equivariant degree. But this equivariant degree (for this larger group) presents strong technical difficulties.

Let us define T as the set $\{(\Gamma x_0, \mu)\colon \mu \in \mathbb{R}\}$ with $x_0 = P^{-1}\bar{a}$. Let S be the set of zeros of $f_h(x, \mu)$, we define $G = S \backslash T$ as the nontrivial solution set. $\bar{G} \backslash G \subset T$ is the bifurcation set and an element of $(x_0, \mu_0) \in \bar{G} \backslash G$ is said to be a bifurcation point. Let $C \subset \bar{G}$ be the connected component of the bifurcation point (x_0, μ_0). Then $C \cap T$ consists of the bifurcation points of the branch C.

We define the collision set as

$$\Psi = \left\{x \in \mathbb{R}^{2(n+1)}\colon x_i = x_j \ (i \neq j)\right\}.$$

Also let $\Lambda_\rho = \{\|\mu\| < \rho\}$ and Ω_ρ be

$$\Omega_\rho = \left\{x \in \mathbb{R}^{2(n+1)}\colon \|x\| < \rho,\ \rho^{-1} < d(x, \Psi)\right\}.$$

Since, for ρ big enough, the set Ω_ρ is a big ball without a small neighborhood of hyperplanes of codimension 2, then the set Ω_ρ is connected.

We say that the component C is admissible whenever it is contained in some set $\Omega_\rho \times \Lambda_\rho$. Otherwise we say that C is inadmissible and this corresponds to the cases where (a): the parameter μ, on the component, goes to infinity, or (b): the norm of x on the component goes to infinity or (c): the component ends at a collision point.

Theorem 15. *If the component C is admissible and the set $C \cap T$ is isolated, then C returns to other bifurcation points $\{(x_1, \mu_1), \ldots, (x_r, \mu_r)\}$ and*

$$\eta_h(x_0, \mu_0) + \cdots + \eta_h(x_r, \mu_r) = 0. \tag{6}$$

Proof. Since C is admissible, we may construct a set $\bar{\Omega} \subset \Omega_\rho \times \Lambda_\rho$ such that $C \subset \Omega$ with $\partial\bar{\Omega} \cap C = \phi$ and such that $f_h(x, \mu)$ is zero on $\partial\bar{\Omega}$ only when $x \in T$. Since $f_h(x, \mu)$ is not zero on $\partial\Omega$ unless $x \in T$, then the degree $\deg(d(x, T) - \varepsilon, f_h; \Omega)$ is well defined. Moreover, as Ω is bounded, we can take ε big enough in such a way that this degree is zero.

By hypothesis $C \cap T$ consists of isolated points. Consequently, taking ε small enough, the points which satisfy $d(x, T) = \varepsilon$ and $f_h(x, \mu) = 0$ in Ω are in the finite and disjoint union of $B_{2\varepsilon}(x_0) \times B_{2\rho}(\mu_0)$ for $(x_0, \mu_0) \in C \cap T$. Hence, by the excision property of the degree we have

$$0 = \deg\bigl(d(x, T) - \varepsilon, f_h; \Omega\bigr) = \sum_{(x_0, \mu_0) \in C \cap T} \deg\bigl(d(x, x_0) - \varepsilon, f_h; B_{2\varepsilon} \times B_{2\rho}\bigr).$$

We conclude the result from the computation of the local degree. □

3.3. Symmetries

The relative equilibria with symmetry $\tilde{D}_h$ are composed of bodies arranged as n/h regular polygons of h sides with some polygons related by reflection. To give a sharper description, let us call an h-gon as the set of positions

$$\left\{re^{i\varphi}e^{k(2\pi i/h)}\colon k = 1, \ldots, h\right\}, \tag{7}$$

and a $2h$-gon as

$$\left\{re^{\pm i\varphi}e^{k(2\pi i/h)}z\colon k = 1, \ldots, h\right\}. \tag{8}$$

Proposition 16. *In a relative equilibrium with symmetries $\tilde{D}_h$ the central body stays on the real axis if $h = 1$ and remains at the origin if $h > 1$. The other bodies satisfy the following arrangements*:

(a) *If n/h is odd, the relative equilibrium has an h-gon* (7) *of bodies with $\varphi = 0$. The remaining bodies form $2h$-gons* (8)*, with $\varphi \in (0, \pi/h)$.*
(b) *If n/h is even, the relative equilibrium has two h-gons* (7) *of bodies, one with $\varphi = 0$ and another with $\varphi = \pi/h$. The remaining bodies form $2h$-gons* (8)*, with $\varphi \in (0, \pi/h)$.*

Proof. Since the central body satisfies the symmetry $x_0 = (\kappa, \kappa)x_0 = \bar{x}_0$, then $x_0 \in \mathbb{R}$. Moreover, as $x_0 = (n/h)(\zeta, \zeta)x_0 = e^{-i2\pi/h}x_0$, then $x_0 = 0$ whenever $h > 1$.

Now, for the remaining bodies, $j \in 1, \ldots, n$, we use the notation $x_j = x_{j+kn}$. Hence these bodies satisfy the relations

$$\text{(a)} \quad x_j = (n/h)(\zeta, \zeta)x_j = e^{-i(2\pi/h)}x_{j+n/h} \quad \text{and} \quad \text{(b)} \quad x_j = (\kappa, \kappa)x_j = \bar{x}_{-j}. \tag{9}$$

By (a), the positions of the bodies are determined only by the bodies with $j \in \mathbb{N} \cap (-n/2h, n/2h]$, and by (b), these are determined by the bodies $j \in \mathbb{N} \cap [0, (n/2h)]$.

From (a) we have an h-gon (7), for each $j \in \{0, (n/2h)\}$. Actually, from (b), we deduce that $\varphi = 0$ for $j = 0$ and $\varphi = \pi/h$ for $j = n/2h$. Now, for each body, $j \in \mathbb{N} \cap (0, (n/2h))$, we have a $2h$-gon (8). Furthermore, since a $2h$-gon has collisions for $\varphi \in \{0, \pi/h\}$, then we can choose $\varphi \in (0, \pi/h)$. □

In order to give examples of the previous descriptions, we shall analyze the cases $\tilde{D}_1$, $\tilde{D}_2$ for n even, and $\tilde{D}_3$ for $n = 6$.

The group $\tilde{D}_1$ is a subgroup of $\tilde{D}_n$, and it is generated by (κ, κ). The relative equilibria with symmetry $\tilde{D}_1$ have the central body on the real axis. The other bodies satisfy the following:

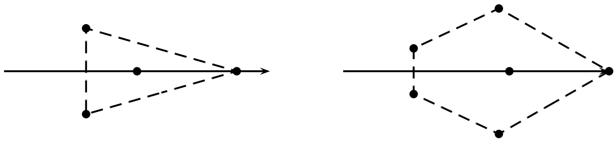
(a) If n is odd. One body is on the real axis. The remaining bodies form symmetric couples with respect to the real axis (see the examples $n = 3$ and $n = 5$).
(b) If n is even. Two bodies are on the real axis without any relation. The remaining bodies form symmetric couples with respect to the real axis (see the examples $n = 4$ and $n = 6$).

The group $\tilde{D}_2$ is generated by (π, π) and (κ, κ), and it is a subgroup of $\tilde{D}_n$ whenever n is even. A relative equilibrium with symmetry $\tilde{D}_2$ has the central body standing still at the origin. The other bodies satisfy the following:

(a) If $n/2$ is odd. One pair of bodies is on the real axis symmetric with respect to the imaginary axis. The remaining bodies form squares symmetric with respect to both axes (see the example $n = 6$).
(b) If $n/2$ is even. One pair of bodies is on the real axis symmetric with respect to the imaginary axis. Another pair of bodies is on the imaginary axis symmetric with respect to the real axis. The remaining bodies form squares which are symmetric with respect to both axes (see the example $n = 4$).

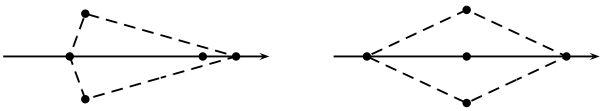
Finally, for the subgroup $\tilde{D}_3$ of $\tilde{D}_6$ we have $n/h = 2$. Therefore, the central body stands still at the origin and the remaining bodies form two triangles without relation, one with $\varphi = 0$ and the other with $\varphi = \pi/3$.

Note that Figs. 1–3 are only an illustration of the possible configurations which may happen. They have to be taken in this perspective, as in the case of the figures in other papers, such as [10]. A precise numerical analysis of the positions of the bodies far from the relative equilibria is outside our present concern. Furthermore, our last proposition is mathematically valid for any μ. However, for the gravitational problem, the masses need to be positive or, at least if one is considering an attraction given by charges, that ω should be positive, since we took its square root. Finally, the spectral analysis and some of the following remarks will give a complement of information and a better justification of our figures.



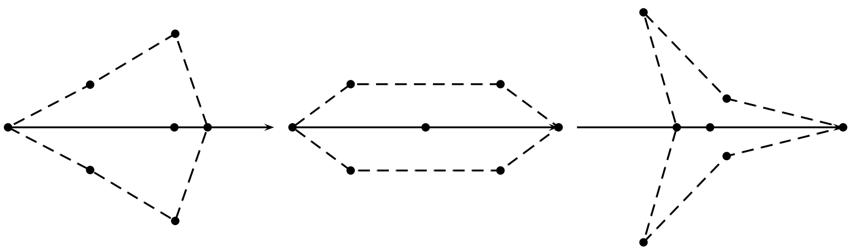
(a) $n = 3$. (b) $n = 5$.

Fig. 1. Symmetries of the group $\tilde{D}_1$.



(a) Symmetries of $\tilde{D}_1$. (b) Symmetries of $\tilde{D}_2$.

Fig. 2. $n = 4$.



(a) Symmetries of $\tilde{D}_1$. (b) Symmetries of $\tilde{D}_2$. (c) Symmetries of $\tilde{D}_3$.

Fig. 3. $n = 6$.

4. Spectral analysis

It is time to calculate explicitly the bifurcation points for the general potential (1). We begin by computing the matrices A_{ij}.

Proposition 17. *Define $\alpha_+ = (\alpha + 1)/2$ and $\alpha_- = (\alpha - 1)/2$, then, for $n \geqslant 3$, we have*

$$A_{00} = \mu(s_1 + \mu + \alpha_- n)I,$$
$$A_{n0} = -\mu(\alpha_- I + \alpha_+ R) \quad \textit{and}$$
$$A_{nn} = (s_1 + \mu)I - \sum_{j=0}^{n-1} A_{nj}.$$

In addition, we have for $j \in \{1, \ldots, n-1\}$ that

$$A_{nj} = \frac{1}{(2\sin(j\zeta/2))^{\alpha+1}}\bigl(-\alpha_- I + \alpha_+ e^{jJ\zeta} R\bigr).$$

Proof. Notice that $\nabla_{x_i}\phi(\|x_i - x_j\|) = -\nabla_{x_j}\phi(\|x_i - x_j\|)$ for $i \neq j$, thus,

$$A_{ij} = \mu_i \mu_j D_{x_j} \nabla_{x_i} \phi\big(\|a_i - a_j\|\big) = -\mu_i \mu_j D^2_{x_i} \phi\big(\|a_i - a_j\|\big).$$

And, for the matrix A_{ii}, we have

$$A_{ii} = (s_1 + \mu)\mu_i I + \sum_{j \neq i} \mu_i \mu_j D^2_{x_i} \phi\big(\|a_i - a_j\|\big) = (s_1 + \mu)\mu_i I - \sum_{j \neq i} A_{ij}.$$

Let us set $a_j = (x_j, y_j)$ and $d_{ij} = \|(x_i, y_i) - (x_j, y_j)\|$, then the function $\phi_\alpha(d_{ij})$ has its matrix of second derivatives

$$D^2 \phi_\alpha(d_{ij}) = \frac{\alpha + 1}{d_{ij}^{\alpha+3}} \begin{pmatrix} (x_i - x_j)^2 & (x_i - x_j)(y_i - y_j) \\ (x_i - x_j)(y_i - y_j) & (y_i - y_j)^2 \end{pmatrix} - \frac{1}{d_{ij}^{\alpha+1}} I.$$

Since the distance from $a_0 = (0, 0)$ to $a_n = (1, 0)$ is $d_{n0} = 1$, then

$$A_{n0} = -\mu \begin{pmatrix} \alpha & 0 \\ 0 & -1 \end{pmatrix} = -\mu(\alpha_- I + \alpha_+ R).$$

Moreover, as $\sum_{j=1}^{n} e^{2jJ\zeta} = 0$ for $\zeta \neq \pi$, or equivalently $n \geqslant 3$, then

$$-\sum_{j=1}^{n} A_{0j} = -\sum_{j=1}^{n} e^{jJ\zeta} A_{n0} e^{-jJ\zeta} = \mu n \alpha_- I.$$

Therefore

$$A_{00} = (s_1 + \mu)\mu I - \sum_{j=1}^{n} A_{0j} = \mu(s_1 + \mu + \alpha_- n) I.$$

It remains only to find the matrix A_{nj} for $j \in \{1, \ldots, n-1\}$. As $a_n = (1, 0)$ and $a_j = (\cos j\zeta, \sin j\zeta)$, then the distance d_{nj} satisfies

$$d_{nj}^2 = (1 - \cos j\zeta)^2 + \sin^2 j\zeta = 4 \sin^2(j\zeta/2).$$

Using the previous results, we have

$$A_{nj} = -\frac{\alpha + 1}{d_{nj}^{\alpha+3}} \begin{pmatrix} (1 - \cos j\zeta)^2 & -(1 - \cos j\zeta)\sin j\zeta \\ -(1 - \cos j\zeta)\sin j\zeta & (\sin j\zeta)^2 \end{pmatrix} + \frac{1}{d_{nj}^{\alpha+1}} I.$$

Now, since $\sin^2 j\zeta = (1 - \cos j\zeta)(1 + \cos j\zeta)$ and $d_{nj}^2 = 2(1 - \cos j\zeta)$, then

$$A_{nj} = \frac{1}{d_{nj}^{\alpha+1}} \left(I - \frac{\alpha + 1}{2} \begin{pmatrix} 1 - \cos j\zeta & -\sin j\zeta \\ -\sin j\zeta & 1 + \cos j\zeta \end{pmatrix} \right).$$

Finally, using $d_{nj} = 2\sin(j\zeta/2)$ we conclude the result. $\square$

It can be seen that the bifurcation points are just the points where $\det B_k$ changes sign. Now we can find explicitly the blocks B_k for the general potential (1).

Proposition 18. *Define s_k, α_k, β_k and γ_k as*

$$s_k = \frac{1}{2^\alpha} \sum_{j=1}^{n-1} \frac{\sin^2(kj\zeta/2)}{\sin^{\alpha+1}(j\zeta/2)},$$

$$\alpha_k = \frac{\alpha_-}{2}(s_{k+1} + s_{k-1}), \qquad \beta_k = \alpha_+(s_k - s_1) \quad \textit{and} \quad \gamma_k = \frac{\alpha_-}{2}(s_{k+1} - s_{k-1}).$$

For $k \in \{2, \ldots, n-2, n\}$, the blocks B_k are

$$B_k = \alpha_+\mu(I + R) + (s_1 + \alpha_k)I - \beta_k R - \gamma_k iJ.$$

Proof. From the definition of B_k and the computation of A_{nn}, we have

$$B_k = (s_1 + \mu)I - A_{n0} + \sum_{j=1}^{n-1} A_{nj}\left(e^{j(ikI+J)\zeta} - I\right).$$

And, from the computation of A_{n0}, we obtain that $B_k = \alpha_+\mu(I + R) + s_1 I + D_k$, with

$$D_k = \sum_{j=1}^{n-1} A_{nj}\left(e^{j(ikI+J)\zeta} - I\right).$$

Now, our problem has been reduced to calculate D_k. Using the explicit computation of A_{nj}, we see that D_k satisfies

$$D_k = \sum_{j=1}^{n-1} \frac{(-\alpha_- I + \alpha_+ e^{jJ\zeta} R)(e^{j(ikI+J)\zeta} - I)}{(2\sin(j\zeta/2))^{\alpha+1}}.$$

The coefficient of the sum can be written as

$$\alpha_-\left(I - e^{j(ikI+J)\zeta}\right) - \alpha_+ R\left(e^{-Jj\zeta} - e^{ijk\zeta}\right).$$

Notice that, using the equalities

$$e^{-(jJ\zeta)} + e^{(jJ\zeta)} = 2I\cos j\zeta \quad \text{and}$$
$$e^{j(ikI+J)\zeta} + e^{-j(ikI+J)\zeta} = 2[I\cos jk\zeta\cos j\zeta + iJ\sin jk\zeta\sin j\zeta],$$

we may cancel terms from the sum D_k for j and $n - j$. In this way, we obtain that the matrix D_k is

$$\sum_{j=1}^{n-1} \frac{\alpha_-(I[1 - \cos kj\zeta\cos j\zeta] - iJ[\sin jk\zeta\sin j\zeta]) - \alpha_+ R[\cos j\zeta - \cos jk\zeta]}{(2\sin(j\zeta/2))^{\alpha+1}}.$$

Hence, we may write D_k as $D_k = \alpha_k I - \beta_k R - \gamma_k i J$ with

$$\alpha_k = \alpha_- \sum \frac{1 - \cos kj\zeta \cos j\zeta}{(2\sin(j\zeta/2))^{\alpha+1}},$$
$$\beta_k = \alpha_+ \sum \frac{\cos j\zeta - \cos jk\zeta}{(2\sin(j\zeta/2))^{\alpha+1}},$$
$$\gamma_k = \alpha_- \sum \frac{\sin jk\zeta \sin j\zeta}{(2\sin(j\zeta/2))^{\alpha+1}}.$$

Finally, we conclude that α_k, β_k and γ_k coincide with the definitions in the proposition from the equalities

$$\alpha_k + \gamma_k = \alpha_- \sum \frac{1 - \cos((k+1)j\zeta)}{(2\sin(j\zeta/2))^{\alpha+1}} = \alpha_- s_{k+1},$$
$$\alpha_k - \gamma_k = \alpha_- \sum \frac{1 - \cos((k-1)j\zeta)}{(2\sin(j\zeta/2))^{\alpha+1}} = \alpha_- s_{k-1},$$
$$\beta_k = \alpha_+ \sum 2\frac{\sin^2(jk\zeta/2) - \sin^2(j\zeta/2)}{(2\sin(j\zeta/2))^{\alpha+1}} = \alpha_+(s_k - s_1). \qquad \square$$

Proposition 19. *For $k \in \{1, n-1\}$, we have that $B_{n-1} = \bar{B}_1$ and*

$$B_1 = \begin{pmatrix} \mu(s_1 + \mu + n\alpha_-) & -(\frac{n}{2})^{1/2}\mu\alpha & -(\frac{n}{2})^{1/2}\mu i \\ -(\frac{n}{2})^{1/2}\mu\alpha & s_1 + \alpha_1 + (\alpha+1)\mu & \alpha_1 i \\ (\frac{n}{2})^{1/2}\mu i & -\alpha_1 i & s_1 + \alpha_1 \end{pmatrix}.$$

Proof. From the proof of the previous proposition, we have that

$$\sum_{j=1}^{n} A_{nj} e^{j(iI+J)\zeta} = \alpha_+(I+R)\mu + (s_1 + \alpha_1)I - \beta_1 R - \gamma_1 i J.$$

And, since $\beta_1 = 0$ and $\alpha_1 = \gamma_1$, then

$$\sum_{j=1}^{n} A_{nj} e^{j(iI+J)\zeta} = \begin{pmatrix} s_1 + \alpha_1 + 2\alpha_+\mu & \alpha_1 i \\ -\alpha_1 i & s_1 + \alpha_1 \end{pmatrix}.$$

Moreover, since $Rv_1 = \bar{v}_1$, then

$$n^{1/2} A_{0n} v_1 = -n^{1/2}\mu(\alpha_- v_1 + \alpha_+ \bar{v}_1) = \mu\left(\frac{n}{2}\right)^{1/2}\begin{pmatrix} -\alpha \\ i \end{pmatrix}.$$

From the definition of B_1 we get the result. Finally, using the computation of B_1, we may prove that $RB_1R = \bar{B}_1$, and then that $B_{n-1} = RB_1R = \bar{B}_1$. $\square$

Clearly, the sums s_k are positive and satisfy $s_k = s_{n+k} = s_{n-k}$. To analyze the bifurcation points, we need the following recursive formula for s_k.

Lemma 20. *Let $\bar{s}_k$ be defined as the sum s_k but with $\alpha - 2$ instead of α. Then, the sums s_k satisfy the recurrence formulae*

$$s_{k+1} - s_k = (2k+1)s_1 - \sum_{h=1}^{k} \bar{s}_h.$$

Proof. We write the sum s_k as

$$2^{\alpha} s_k = \sum_{j=1}^{n-1} \frac{1}{\sin^{\alpha-1}(j\zeta/2)} \frac{1 - \cos(kj\zeta)}{1 - \cos(j\zeta)}.$$

Using geometric series, we have

$$\frac{1 - \cos(kj\zeta)}{1 - \cos(j\zeta)} = \frac{1 - e^{ijk\zeta}}{1 - e^{ij\zeta}} \frac{1 - e^{-ijk\zeta}}{1 - e^{-ij\zeta}} = \sum_{l=0}^{k-1} \sum_{m=0}^{k-1} e^{ij(l-m)\zeta}.$$

Now, we may cancel common terms from s_{k+1} and s_k as

$$2^{\alpha}(s_{k+1} - s_k) = \sum_{j=1}^{n-1} \frac{1}{\sin^{\alpha-1}(j\zeta/2)} \sum_{h=-k}^{k} e^{ijh\zeta}.$$

Finally, since

$$\sum_{h=-k}^{k} e^{ijh\zeta} = \sum_{h=-k}^{k} \cos jh\zeta = (2k+1) - 4\sum_{h=1}^{k} \sin^2(jh\zeta/2),$$

then

$$s_{k+1} - s_k = (2k+1)s_1 - \sum_{h=1}^{k} \sum_{j=1}^{n-1} \frac{\sin^2(hj\zeta/2)}{2^{\alpha-2}\sin^{\alpha-1}(j\zeta/2)} = (2k+1)s_1 - \sum_{h=1}^{k} \bar{s}_h. \qquad \square$$

The idea of using geometric series is taken from [1], where it is used to calculate s_k for the vortex case $\alpha = 1$. Iterating this result we obtain the equalities

$$s_{k+1} - 2s_k + s_{k-1} = 2s_1 - \bar{s}_k \tag{10}$$

and

$$s_k = \sum_{l=0}^{k-1} (s_{l+1} - s_l) = \sum_{l=0}^{k-1} \left((2l+1)s_1 - \sum_{h=1}^{l} \bar{s}_h \right) = k^2 s_1 - \sum_{l=1}^{k-1} l\bar{s}_{k-l}. \tag{11}$$

4.1. General potential

Now, we only need to find the bifurcation points, that is, the points where $\sigma_k(\mu)$ changes sign for the general potential (1). For $k=n$, we have that $\beta_n = -\alpha_+ s_1$ and $\alpha_n = \alpha_- s_1$, then $e_1^T B_n e_1 = (\alpha+1)(\mu+s_1)$ and

$$\sigma_n(\mu) = sgn(\mu + s_1).$$

Proposition 21. *The sign of σ_1 is*

$$\sigma_1(\mu) = sgn\big(b_1\mu(\mu+s_1)(\mu-\mu_1)\big),$$

where $\mu_1 = -a_1/b_1$, with

$$a_1 = (s_1 + 2\alpha_1)(2s_1 + n\alpha - n) \quad \textit{and} \quad b_1 = (\alpha+1)(2s_1 + 2\alpha_1 - n).$$

Proof. We get the result from the fact that $\det B_1$ can be factored as follows

$$\begin{aligned}\frac{2\det B_1}{\mu(\mu+s_1)} &= \mu(\alpha+1)(2s_1+2\alpha_1-n) + (s_1+2\alpha_1)(2s_1+n\alpha-n)\\ &= b_1(\mu-\mu_1). \qquad \square\end{aligned}$$

Remark 22. Notice that σ_1 and σ_n change sign at $-s_1$, then $\eta_1(-s_1)=0$ and $\eta_n(-s_1)=\pm 2$. Nevertheless, there are two explicit bifurcations at $-s_1$ one with symmetry $\tilde{D}_1$ and another one with $\tilde{D}_n$. Indeed, the bifurcation with symmetry $\tilde{D}_1$ is made of the translations of $\bar{a}$, $(0+r, e^{i\zeta}+r, \dots, e^{in\zeta}+r)$, with $\omega = 0$. The bifurcation with symmetry $\tilde{D}_n$ is made of the homotheties of $\bar{a}$, $(0, e^{i\zeta}r, \dots, e^{in\zeta}r)$, with $\omega = 0$.

In addition, since σ_1 changes sign at $\mu = 0$, then $\eta_1(0) = \pm 2$. Therefore, there must be a bifurcation, with symmetry $\tilde{D}_1$, at $\mu = 0$. As the central body has mass zero, $\mu = 0$, then the bifurcation has no physical meaning since it is made of the solutions $(r, e^{i\zeta}, \dots, e^{in\zeta})$, with $\mu = 0$.

Proposition 23. *For $k \in \{2, \dots, [n/2]\}$, the signs of σ_k are*

$$\sigma_k(\mu) = sgn(\mu - \mu_k),$$

where $\mu_k = -a_k/b_k$, with

$$a_k = (s_1+\alpha_k)^2 - \gamma_k^2 - \beta_k^2 \quad \textit{and} \quad b_k = (\alpha+1)(s_1+\alpha_k+\beta_k).$$

Proof. For $k = n/2$, we have that $\gamma_{n/2} = 0$ and

$$e_1^T B_{n/2} e_1 = s_1 + \alpha_{n/2} - \beta_{n/2} + (\alpha+1)\mu.$$

For $k \notin \{1, n/2, n\}$, the determinant of B_k is

$$\det B_k = \big(\alpha_+\mu + (s_1+\alpha_k)\big)^2 - \gamma_k^2 - (\alpha_+\mu - \beta_k)^2 = b_k\mu + a_k.$$

From the definitions of α_k and β_k, we have that

$$s_1+\alpha_k+\beta_k=s_k+(\alpha_-/2)(s_{k+1}+2s_k+s_{k-1}-2s_1).$$

Using the equality (10) and the fact that $4s_k-\bar{s}_k$ is positive, we get the inequality $s_{k+1}+2s_k+s_{k-1}>2s_1$. Consequently, the factor b_k is positive and we may conclude the result. $\square$

From the discussion in the previous remark, the true bifurcations are found at μ_k for $k\in\{1,\ldots,[n/2]\}$. From the bifurcation theorem we have the following:

Theorem 24. *For $k\in\{1,\ldots,[n/2]\}$, let h be the maximum common divisor of k and n. If μ_k is different from $-s_1$, 0 and μ_j for the other $j\in[1,n/2]\cap h\mathbb{N}$, then, from μ_k, there is a global bifurcation of relative equilibria with maximal symmetry $\tilde{D}_h$.*

By maximal symmetry $\tilde{D}_h$ we mean that the local branch has symmetry $\tilde{D}_h$ but not for a bigger group $\tilde{D}_p$.

By global bifurcation we mean that, whenever the branch is admissible, the branch returns to other bifurcation points and the sum of the local degrees at these bifurcation points is zero. The branch is inadmissible when the parameter or the norm goes to infinity, or when the branch ends in a collision solution.

Remark 25. Notice that these results are applicable only for $n\geqslant 3$, since the irreducible representations of the definition (4) are not consistent for $n=2$. Nevertheless, the case $n=2$ was analyzed in the same spirit in a previous remark. For instance, we have proved that there is a bifurcation of relative equilibria with symmetry $\tilde{D}_1$ from $\mu_1=-(2\alpha+s_1)/(\alpha+1)$. Also we did calculate, for the vortex problem, that $\mu_1=-5/4$ and, for the body problem, $\mu_1=-17/12$.

4.2. $(n+1)$-vortex potential

Here we give a short description of the bifurcation points for the vortex problem, since, in this case, we can calculate explicitly the bifurcation points μ_k.

Proposition 26. *For $\alpha=1$, we have that*

$$s_k=k(n-k)/2.$$

Proof. For $\alpha=1$, we have $s_1=(n-1)/2$. In addition, we may calculate $\bar{s}_k$ as

$$\bar{s}_k=2\sum_{j=1}^{n-1}\sin^2(kj\zeta/2)=\sum_{j=1}^{n-1}\bigl(1-\cos(kj\zeta)\bigr)=n.$$

Therefore, from the formula (11), we have that

$$s_k=k^2(n-1)/2-n\sum_{l=1}^{k-1}l=k(n-k)/2. \qquad \square$$

From the definitions with $\alpha=1$, we have $\alpha_-=0$, $\alpha_+=1$, $\alpha_k=0$, $\gamma_k=0$ and $\beta_k=s_k-s_1$. Since $\mu_k=s_k/2-s_1$, for $k\in\{2,\ldots,[n/2]\}$, then

$$\mu_k=\bigl(-k^2+nk-2n+2\bigr)/4.$$

And, for $k=1$, we have

$$\mu_1 = s_1^2 = (n-1)^2/4.$$

Consequently, the bifurcation point $\mu_2 = -1/2$ is always negative and $\mu_3 = (n-7)/4$ is positive only for $n \geqslant 8$. Moreover, since μ_k is increasing in n for $k \geqslant 3$, then μ_k is always positive for $k \geqslant 4$. Notice also that the bifurcation points μ_k are increasing in k for $k \in \{2, \dots, [n/2]\}$, and, as a consequence, the μ_k are different.

Theorem 27. *For $n \geqslant 3$, and each $k \in \{1, \dots, [n/2]\}$, the polygonal relative equilibrium has a global bifurcation of relative equilibria from μ_k with maximal symmetry $\tilde{D}_h$.*

The existence of the local bifurcation was proved before in the article [10], with a normal form method.

4.3. $(n+1)$-body potential

Notice that, for the $(n+1)$-body problem, the equations have a physical meaning only for $\mu \geqslant 0$. Given that we cannot calculate explicitly the sums s_k in this case, we shall give an asymptotic computation of the sums s_k and of the bifurcation points μ_k.

Proposition 28. *For n big enough, the bifurcation point μ_1 is negative and μ_k is positive for $k \geqslant 2$.*

Proof. For the $(n+1)$-body problem $\alpha = 2$. From the definitions, we have in this case $\alpha_- = 1/2$, $\alpha_+ = 3/2$, $\alpha_k = (s_{k+1} + s_{k-1})/4$, $\beta_k = 3(s_k - s_1)/2$ and $\gamma_k = (s_{k+1} - s_{k-1})/4$.

Using integral estimates, it can be easily seen that $s_1/n \to \infty$ and that $\bar{s}_k/n$ is finite when n goes to infinity. Therefore, from the formula (11), we have the limits $s_k/s_1 \to k^2$, when n goes to infinity, see [13].

We have, for $k \geqslant 2$, that $\beta_k/s_1 \to 3(k^2-1)/2$, $\alpha_k/s_1 \to (k^2+1)/2$ and $\gamma_k/s_1 \to k$, when $n \to \infty$. Therefore, from the definitions of a_k and b_k, we obtain the limits $b_k/s_1 \to 6k^2$ and

$$a_k/s_1^2 = (1 + \alpha_k/s_1)^2 - (\beta_k/s_1)^2 - (\gamma_k/s_1)^2 \to -k^2\left(2k^2 - 5\right).$$

Consequently, the result follows from the fact that μ_k/s_1 converges to the positive limit $(2k^2 - 5)$ for $k \geqslant 2$.

For $k = 1$, we have that $\alpha_1/s_1 \to 1$, then we obtain the result from

$$\mu_1/s_1 = -\frac{(s_1 + 2\alpha_1)(2s_1 + n)}{3(2s_1 + 2\alpha_1 - n)} \to -1/2. \qquad \square$$

In [10], the bifurcation of the local branch from μ_k is proved for the $(n+1)$-body problem.

Remark 29. Given the numerical evidence of μ_k, for instance see [10], it seems that $\mu_1 \geqslant 0$ for $n \in \{3, 4, 5, 6\}$, $\mu_2 \geqslant 0$ for $n \geqslant 10$ and $\mu_k \geqslant 0$ for every $k \geqslant 3$. The numerical evidence also suggests that the μ_k are increasing for $k \in \{2, \dots, [n/2]\}$. This is true at least in the limit when $n \to \infty$, because $(\mu_{k+1} - \mu_k)/s_1$ converges to the positive limit $(2k+1)/3$.

Theorem 30. *Assuming the numerical evidence of the previous remark, from μ_1 for $n \in \{3, 4, 5, 6\}$, from μ_2 for $n \geqslant 10$, and from μ_k for each $k \in \{3, \dots, [n/2]\}$, the polygonal relative equilibrium has a global bifurcation of relative equilibria with maximal symmetry $\tilde{D}_h$.*

5. dNLS

The dNLS equations are

$$i\dot{q}_j = h\big(\|q_j\|^2\big)q_j + (q_{j+1} - 2q_j + q_{j-1}),$$

where $q_j \in \mathbb{C}$ represents the oscillator and h is the nonlinear potential. We wish to study a finite circular lattice, that is, a lattice of oscillators for $j \in \{1, \dots, n\}$, with periodic conditions $q_j = q_{j+n}$.

The solutions of the form $q_j = e^{\omega t i}x_j$, with x_j constant, are called relative equilibria. In order to obtain the amplitude as a parameter, we need to change coordinates, with $q_j = \mu e^{\omega t i}x_j$. In this manner, we have that the values x_j form a relative equilibrium when

$$-\omega x_j = h\big(|\mu x_j|^2\big)x_j + (x_{j+1} - 2x_j + x_{j-1}).$$

Remark 31. Given that the lattice is integrable for $n = 1$ and $n = 2$, we shall look for bifurcation of relative equilibria for $n \geqslant 3$. Actually, according to [3], it is possible to find all the bifurcation diagrams of the relative equilibria for $n \leqslant 4$. Notice that the relative equilibria are known as breathers when they are localized. For $n = 3$, see [8].

The starting point is a relative equilibrium which looks like a rotating wave and is the equivalent of the polygonal relative equilibrium in the n-body problem. We give next a condition which needs to be satisfied by the potential for the existence of this rotating wave.

Proposition 32. *Define $a_j = e^{ij\zeta}$, with $\zeta = 2\pi/n$, then $\bar{a} = (a_1, \dots, a_n)$ is a relative equilibrium if*

$$\omega = 4\sin^2(\zeta/2) - h\big(\mu^2\big).$$

Proof. Since $a_{j+1} - 2a_j + a_{j-1} = -4\sin^2(\zeta/2)a_j$, then

$$V_{x_j}(\bar{a}) = \big(\omega + h\big(\mu^2\big) - 4\sin^2(\zeta/2)\big)a_j. \qquad \square$$

Remark 33. Note that the existence of the rotating wave is determined by a non-homogeneous relation between the amplitude μ and the frequency ω. This is different from the n-body problem, where the existence of the relative equilibrium is determined by a homogeneous relation.

In order to show the similarities with the n-body problem we change to real coordinates. Let $x = (x_1, \dots, x_n) \in \mathbb{R}^{2n}$ be the vector of positions, then the relative equilibria are critical points of the potential

$$V(x) = \frac{1}{2}\sum_{j=1}^{n}\big\{H(x_j, \mu) - |x_{j+1} - x_j|^2\big\},$$

where $x_j = x_{j+n}$ and $H(x, \mu)$ is a function such that $\nabla H(x) = \omega x + h(|\mu x|^2)x$.

From the point of view of the symmetries, there is practically no difference with the definitions of the n-body problem (2). The unique difference is in the fact that we are not including the coordinate x_0 of the n-body problem. So, in this case, the group D_n acts on $\mathbb{R}^{2n}$ as $\rho(\gamma)(x_1, \dots, x_n) = (x_{\gamma(1)}, \dots, x_{\gamma(n)})$ and the group $O(2) = S^1 \cup \kappa S^1$ in a similar way. The fact that the gradient ∇V is D_n-equivariant follows from the periodicity conditions $x_j = x_{j+n}$. Moreover, it is well known that the potential is invariant when we rotate the phases of all oscillators, so the gradient ∇V is $O(2)$-equivariant.

As a consequence, we may adapt the results of Sections 2 and 3. Actually, as in Proposition 7, in this case the blocks are given by

$$B_k = \sum_{j=1}^{n} A_{nj} e^{j(ikI+J)\zeta} \tag{12}$$

for $k \in \{1, \ldots, n/2, n\}$, and the signs of $\sigma(\mu)$ are defined as before in (5). Furthermore, since, in this case, there is no collision points, then the bifurcation is inadmissible only when the parameter μ or the norm of the branch goes to infinity.

We wish to describe briefly the meaning of the symmetries $\tilde{D}_h$ (9) for the dNLS equations. Due to $x_j = e^{-i(2\pi/h)} x_{j+n/h}$, then the solutions look like rotating waves composed of h identical waves, each one formed by $\bar{n} = n/h$ oscillators which satisfy the reflection symmetry $x_j = \bar{x}_{\bar{n}-j}$. An example of relative equilibria with symmetry $\tilde{D}_h$ is

$$x_j = \left(1 + \varepsilon \sin^2 j(\pi/\bar{n})\right) e^{i(2\pi/h)}.$$

Given that most of the work is already done, we shall focus our attention on finding the bifurcation points.

5.1. General potential

Again, the first step is to find the submatrices A_{ij} of $D^2 V(x)$ at a_j.

Proposition 34. *The submatrices A_{nj} are $A_{nj} = I$ for $j \in \{1, n-1\}$, $A_{nj} = 0$ for $j \notin \{1, n-1, n\}$ and*

$$A_{nn} = (-2\cos\zeta) I + 2\mu^2 h'\left(\mu^2\right) diag(1, 0).$$

Proof. As the coupling is linear and only between adjacent oscillators, then $A_{nj} = I$ for $j \in \{1, n-1\}$, $A_{nj} = 0$ for $j \notin \{1, n-1, n\}$ and $A_{nn} = D^2 H(a_n) - 2I$.

Let $x_0 = (x, y)$, since $\nabla H(x_0) = \omega x_0 + h(|\mu x_0|^2) x_0$, then

$$D^2 H(x_0) = \left(\omega + h\left(|\mu x_0|^2\right)\right) I + 2\mu^2 h'\left(|\mu x_0|^2\right) \begin{pmatrix} x^2 & xy \\ xy & y^2 \end{pmatrix}.$$

Since $\bar{a}$ is an equilibrium when $\omega + h(\mu^2) = 4\sin^2(\zeta/2)$, then at $a_n = (1, 0)$ we have

$$D^2 H(a_n) = 4\sin^2(\zeta/2) I + 2\mu^2 h'\left(|\mu|^2\right) diag(1, 0).$$

Hence, we conclude the result from the equality $4\sin^2(\zeta/2) - 2 = -2\cos\zeta$. $\square$

Now we may calculate the blocks B_k from (12).

Proposition 35. *Define α_k and γ_k as*

$$\alpha_k = 4\cos\zeta \sin^2 k\zeta/2 \quad \text{and} \quad \gamma_k = 2\sin k\zeta \sin\zeta.$$

Then, the blocks B_k are

$$B_k = -\alpha_k I + \gamma_k (iJ) + 2\mu^2 h'\left(|\mu|^2\right) diag(1, 0).$$

Proof. Using the explicit computation of A_{nj}, we have

$$B_k = \left(e^{(ikI+J)\zeta} + e^{-(ikI+J)\zeta}\right) + (-2\cos\zeta)I + 2\mu^2 h'\left(\mu^2\right) diag(1,0).$$

Then, from the equalities $2\cos\zeta(\cos k\zeta - 1) = -\alpha_k$ and

$$e^{(ikI+J)\zeta} + e^{-(ikI+J)\zeta} = (2\cos k\zeta \cos\zeta)I + (2\sin k\zeta \sin\zeta)iJ,$$

we obtain the form of B_k. $\square$

Now, it remains to find the bifurcation points. Since $\alpha_n = 0$, then we have $e_1^T B_n e_1 = 2\mu^2 h'(\mu^2)$ and

$$\sigma_n = sgn\, h'\left(\mu^2\right).$$

Since, for $n = 3$, we have $\alpha_1 = -\gamma_1 < 0$, then $\sigma_1 = sgn\, h'(\mu^2)$. As for $n = 4$, we have $\alpha_k = 0$ and $\gamma_1 \neq 0$, then $\sigma_1 = -1$ and $\sigma_2 = sgn(h'(\mu^2))$. Given that, in our examples, $h'(\mu^2)$ does not change sign, there are no bifurcation points for $n = 3, 4$.

Consequently, we shall focus our attention only on the cases $n \geqslant 5$, for $k \in \{1, \ldots, n/2\}$, where we can assume

$$\alpha_k = 4\cos\zeta \sin^2 k\zeta/2 \geqslant 0.$$

Proposition 36. *For $n \geqslant 5$ and $k \in \{1, \ldots, n/2\}$, the sign of σ_k can change sign only for the solutions of $\mu^2 h'(\mu^2) = \delta_k$, with*

$$\delta_k = \left(\alpha_k^2 - \gamma_k^2\right)/(2\alpha_k).$$

Moreover, we have $\delta_1 < 0$, $\delta_2 = 0$ and $\delta_k > 0$ for $k \in [3, n/2] \cap \mathbb{N}$.

Proof. For $k \in [1, n/2) \cap \mathbb{N}$, we have

$$\begin{aligned} \det B_k &= \alpha_k^2 - \gamma_k^2 - 2\alpha_k \mu^2 h'\left(\mu^2\right) \\ &= 2\alpha_k\left(\delta_k - \mu^2 h'\left(\mu^2\right)\right). \end{aligned}$$

Since $\alpha_k > 0$, then $\sigma_k = sgn(\delta_k - \mu^2 h'(\mu^2))$.

When $k = n/2$, we have $\gamma_{n/2} = 0$, then $e_1^T B_{n/2} e_1 = -\alpha_{n/2} + 2\mu^2 h'(\mu^2)$. Thus, $\sigma_{n/2} = sgn(\mu^2 h'(\mu^2) - \alpha_{n/2})$ changes sign only for the solutions of $\mu^2 h'(\mu^2) = \alpha_{n/2}/2 = \delta_{n/2}$.

Finally, since δ_k has the sign of

$$\alpha_k^2 - \gamma_k^2 = 16\left(\sin^2 k\zeta/2 - \sin^2\zeta\right)\sin^2 k\zeta/2,$$

then δ_k has the sign of $\sin^2 k\zeta/2 - \sin^2\zeta$. $\square$

From the bifurcation Theorems 14 and 15, we have the following result.

Theorem 37. *For each simple solution of $\mu_k^2 h'(\mu_k^2) = \delta_k$, from the amplitude μ_k we have a global bifurcation of relative equilibria with symmetry $\tilde{D}_h$, where h is the maximum common divisor of k and n.*

Remark 38. Actually, we may analyze more complex lattices whenever we preserve the symmetries. For instance, we may consider nonlinear coupling and coupling with distant oscillators.

Now, we wish to give two typical examples.

5.2. *The Schrödinger cubic potential*

For the cubic Schrödinger potential, we need to set $h(x) = x$. In this case $h'(\mu^2) = 1$ and $\sigma_n = 1$.

Then, for $n \geqslant 5$, the sign of $\sigma_k(\mu)$ changes only when $\mu_k = \sqrt{\delta_k}$, if δ_k is positive. As we have proved before that δ_k is positive when $n \geqslant 6$, for $k \in \{3, \ldots, [n/2]\}$, and, since the numbers δ_k are increasing in k, then the μ_k are increasing for $k \in \{3, \ldots, [n/2]\}$.

Theorem 39. *For the cubic Schrödinger potential, for $n \in \{6, 7, \ldots\}$, for each $k \in \{3, \ldots, [n/2]\}$ there is a global bifurcation of relative equilibria with maximal symmetry $\tilde{D}_h$ from the amplitude $\sqrt{\delta_k}$.*

5.3. *A saturable potential*

For a saturable potential, we need to set $h = (1+x)^{-1}$. In this case, $h'(\mu^2) = -(1+\mu^2)^{-2}$, $\sigma_n = -1$ and

$$\mu^2 h'(\mu^2) = -\mu^2(1+\mu^2)^{-2}$$

is a function with range $(-1/4, 0)$ and a single minimum at $\mu^2 = 1$. Therefore, there are two zeros, $\mu_- \in (0, 1)$ and $\mu_+ \in (1, \infty)$, of the equation $\mu^2 h'(\mu^2) = \delta_k$, when $\delta_k \in (-1/4, 0)$.

Since we have proved before that $\delta_k \geqslant 0$ for $k \geqslant 2$, there is no bifurcation for $k \geqslant 2$ and it remains only to analyze the case $k = 1$. Since

$$\delta_1 = 2\left(\sin^2 \zeta/2 - \sin^2 \zeta\right)/\cos\zeta \to 0^-$$

when $n \to \infty$, then $\delta_1 \in (-1/4, 0)$ if n is big enough. Indeed, we obtain numerically that $\delta_1 \in (-1/4, 0)$ for $n \geqslant 16$.

Theorem 40. *For the lattice with saturable potential, for $n \geqslant 16$, from the amplitudes $\mu_- \in (0, 1)$ and $\mu_+ \in (1, \infty)$ there is a bifurcation of relative equilibria with maximal symmetry $\tilde{D}_1$.*

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